Open and other kinds of extensions over zero-dimensional local compactifications

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Abstract

Generalizing a theorem of Ph. Dwinger [7], we describe the partially ordered set of all (up to equivalence) zero-dimensional locally compact Hausdorff extensions of a zero-dimensional Hausdorff space. Using this description, we find the necessary and sufficient conditions which has to satisfy a map between two zero-dimensional Hausdorff spaces in order to have some kind of extension over arbitrary given in advance Hausdorff zero-dimensional local compactifications of these spaces; we regard the following kinds of extensions: continuous, open, quasi-open, skeletal, perfect, injective, surjective. In this way we generalize some classical results of B. Banaschewski [1] about the maximal zero-dimensional Hausdorff compactification. Extending a recent theorem of G. Bezhanishvili [2], we describe the local proximities corresponding to the zero-dimensional Hausdorff local compactifications.

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Introduction

In [1], B. Banaschewski proved that every zero-dimensional Hausdorff space X has a zero-dimensional Hausdorff compactification $\beta_0 X$ with the following remarkable property: every continuous map $f : X \longrightarrow Y$, where Y is a zero-dimensional Hausdorff compact space, can be extended to a continuous map $\beta_0 f : \beta_0 X \longrightarrow Y$; in

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particular, $\beta_0 X$ is the maximal zero-dimensional Hausdorff compactification of X. As far as I know, there are no descriptions of the maps f for which the extension $\beta_0 f$ is open or quasi-open. In this paper we solve the following more general problem: let $f : X \longrightarrow Y$ be a map between two zero-dimensional Hausdorff spaces and (lX, l_X), (lY, l_Y) be Hausdorff zero-dimensional locally compact extensions of X and Y, respectively; find the necessary and sufficient conditions which has to satisfy the map f in order to have an "extension" $g : lX \longrightarrow lY$ (i.e. $g \circ l_X = l_Y \circ f$) which is a map with some special properties (we regard the following properties: continuous, open, perfect, quasi-open, skeletal, injective, surjective). In [10], S. Leader solved such a problem for continuous extensions over Hausdorff local compactifications (= locally compact extensions) using the language of *local proximities* (the later, as he showed, are in a bijective correspondence (preserving the order) with the Hausdorff local compactifications regarded up to equivalence). Hence, if one can describe the local proximities which correspond to zero-dimensional Hausdorff local compactifications then the above problem will be solved for continuous extensions. Recently, G. Bezhanishvili [2], solving an old problem of L. Esakia, described the *Efremovič proximities* which correspond (in the sense of the famous Smirnov Compactification Theorem [16]) to the zero-dimensional Hausdorff compactifications (and called them *zero-dimensional Efremovič proximities*). We extend here his result to the Leader's local proximities, i.e. we describe the local proximities which correspond to the Hausdorff zero-dimensional local compactifications and call them *zero-dimensional local proximities* (see Theorem 3.2). We do not use, however, these zero-dimensional local proximities for solving our problem. We introduce a simpler notion (namely, the *admissibe ZLB-algebra*) for doing this. Ph. Dwinger [7] proved, using Stone Duality Theorem [17], that the ordered set of all, up to equivalence, zero-dimensional Hausdorff compactifications of a zero-dimensional Hausdorff space is isomorphic to the ordered by inclusion set of all *Boolean bases* of X (i.e. of those bases of X which are Boolean subalgebras of the Boolean algebra $CO(X)$ of all clopen (= closed and open) subsets of X). This description is much simpler than that by Efremovič proximities. It was rediscovered by K. D. Magill Jr. and J. A. Glasenapp [11] and applied very successfully to the study of the poset of all, up to equivalence, zero-dimensional Hausdorff compactifications of a zero-dimensional Hausdorff space. We extend the cited above Dwinger Theorem [7] to the zero-dimensional Hausdorff *local compactifications* (see Theorem 2.3 below) with the help of our generalization of the Stone Duality Theorem proved in [5] and the notion of "admissible ZLB-algebra" which we introduce here. We obtain the solution of the problem formulated above in the language of the admissible ZLB-algebras (see Theorem 4.8). As a corollary, we characterize the maps $f : X \longrightarrow Y$ between two Hausdorff zero-dimensional spaces X and Y for which the extension $\beta_0 f : \beta_o X \longrightarrow \beta_0 Y$ is open or quasi-open (see Corollary 4.9). Of course, one can pass from admissible ZLB-algebras to zero-dimensional local proximities and conversely (see Theorem 3.4 below; it generalizes an analogous result about the connection between Boolean bases and zero-dimensional Efremovič proximities obtained in [2]).

We now fix the notations.

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if f is a morphism of $\mathcal{C}$ with domain X and codomain Y. By $Id_{\mathcal{C}}$ we denote the identity functor on the category $\mathcal{C}$.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct. Since we follow Johnstone's terminology from [9], we will use the term *pseudolattice* for a poset having all finite non-empty meets and joins; the pseudolattices with a bottom will be called $\{0\}$-*pseudolattices*. If B is a Boolean algebra then we denote by $Ult(B)$ the set of all ultrafilters in B.

If X is a set then we denote the power set of X by $P(X)$; the identity function on X is denoted by id_X.

If (X, τ) is a topological space and M is a subset of X, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of M in (X, τ) and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of M in (X, τ).

The closed maps and the open maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is *perfect* if it is closed and compact (i.e. point inverses are compact sets).

For all notions and notations not defined here see [7, 8, 9, 14].

1 Preliminaries

We will need some of our results from [5] concerning the extension of the Stone Duality Theorem to the category **ZLC** of all locally compact zero-dimensional Hausdorff spaces and all continuous maps between them.

Recall that if $(A, \leq)$ is a poset and $B \subseteq A$ then B is said to be a *dense subset of* A if for any $a \in A \setminus \{0\}$ there exists $b \in B \setminus \{0\}$ such that $b \leq a$.

Definition 1.1 ([5]) A pair (A, I), where A is a Boolean algebra and I is an ideal of A (possibly non proper) which is dense in A, is called a *local Boolean algebra* (abbreviated as LBA). Two LBAs (A, I) and (B, J) are said to be *LBA-isomorphic* (or, simply, *isomorphic*) if there exists a Boolean isomorphism $\varphi : A \longrightarrow B$ such that $\varphi(I) = J$.

Let A be a distributive $\{0\}$-pseudolattice and $Idl(A)$ be the frame of all ideals of A. If $J \in Idl(A)$ then we will write $\neg_A J$ (or simply $\neg J$) for the pseudocomplement of J in $Idl(A)$ (i.e. $\neg J = \bigvee\{I \in Idl(A) \mid I \wedge J = \{0\}\}$). Recall that an ideal J of A is called *simple* (Stone [17]) if $J \vee \neg J = A$ (i.e. J has a complement in $Idl(A)$). As it is proved in [17], the set $Si(A)$ of all simple ideals of A is a Boolean algebra with respect to the lattice operations in $Idl(A)$.

Definition 1.2 ([5]) An LBA (B, I) is called a *ZLB-algebra* (briefly, *ZLBA*) if, for every $J \in Si(I)$, the join $\bigvee_B J (= \bigvee_B\{a \mid a \in J\})$ exists.

Let **ZLBA** be the category whose objects are all ZLBAs and whose morphisms are all functions $\varphi : (B, I) \longrightarrow (B_1, I_1)$ between the objects of **ZLBA** such that $\varphi : B \longrightarrow B_1$ is a Boolean homomorphism satisfying the following condition:

(ZLBA) For every $b \in I_1$ there exists $a \in I$ such that $b \leq \varphi(a)$;

let the composition between the morphisms of **ZLBA** be the usual composition between functions, and the **ZLBA**-identities be the identity functions.

Example 1.3 ([5]) Let B be a Boolean algebra. Then the pair (B, B) is a ZLBA.

Notations 1.4 Let X be a topological space. We will denote by $CO(X)$ the set of all clopen (= closed and open) subsets of X, and by $CK(X)$ the set of all clopen compact subsets of X. For every $x \in X$, we set $u_x^{CO(X)} = \{F \in CO(X) \mid x \in F\}$. When there is no ambiguity, we will write "u_x^C" instead of "$u_x^{CO(X)}$".

The next assertion follows from the results obtained in [15, 5].

Proposition 1.5 *Let (A, I) be a ZLBA. Set $X = \{u \in Ult(A) \mid u \cap I \neq \emptyset\}$. Set, for every $a \in A$, $\lambda_A^C(a) = \{u \in X \mid a \in u\}$. Let τ be the topology on X having as an open base the family $\{\lambda_A^C(a) \mid a \in I\}$. Then (X, τ) is a zero-dimensional locally compact Hausdorff space, $\lambda_A^C(A) = CO(X)$, $\lambda_A^C(I) = CK(X)$ and $\lambda_A^C : A \longrightarrow CO(X)$ is a Boolean isomorphism; hence, $\lambda_A^C : (A, I) \longrightarrow (CO(X), CK(X))$ is a* **ZLBA**-*isomorphism. We set $\Theta^a(A, I) = (X, \tau)$.*

Theorem 1.6 ([5]) *The category* **ZLC** *is dually equivalent to the category* **ZLBA**. *In more details, let Θ^a : $\mathbf{ZLBA} \longrightarrow \mathbf{ZLC}$ and $\Theta^t : \mathbf{ZLC} \longrightarrow \mathbf{ZLBA}$ be two contravariant functors defined as follows: for every $X \in |\mathbf{ZLC}|$, we set $\Theta^t(X) = (CO(X), CK(X))$, and for every $f \in \mathbf{ZLC}(X, Y)$, $\Theta^t(f) : \Theta^t(Y) \longrightarrow \Theta^t(X)$ is defined by the formula $\Theta^t(f)(G) = f^{-1}(G)$, where $G \in CO(Y)$; for the definition of $\Theta^a(B, I)$, where (B, I) is a ZLBA, see 1.5; for every $\varphi \in \mathbf{ZLBA}((B, I), (B_1, J))$, $\Theta^a(\varphi) : \Theta^a(B_1, J) \longrightarrow \Theta^a(B, I)$ is given by the formula $\Theta^a(\varphi)(u') = \varphi^{-1}(u')$, $\forall u' \in \Theta^a(B_1, J)$; then $t^C : Id_{\mathbf{ZLC}} \longrightarrow \Theta^a \circ \Theta^t$, where $t^C(X) = t_X^C$, $\forall X \in |\mathbf{ZLC}|$ and $t_X^C(x) = u_x^C$, for every $x \in X$, is a natural isomorphism (hence, in particular, $t_X^C : X \longrightarrow \Theta^a(\Theta^t(X))$ is a homeomorphism for every $X \in |\mathbf{ZLC}|$); also, $\lambda^C : Id_{\mathbf{ZLBA}} \longrightarrow \Theta^t \circ \Theta^a$, where $\lambda^C(B, I) = \lambda_B^C$, $\forall (B, I) \in |\mathbf{ZLBA}|$, is a natural isomorphism.*

Finally, we will recall some definitions and facts from the theory of extensions of topological spaces, as well as the fundamental Leader's Local Compactification Theorem [10].

Let X be a Tychonoff space. We will denote by $\mathfrak{L}(X)$ the set of all, up to equivalence, locally compact Hausdorff extensions of X (recall that two (locally compact Hausdorff) extensions (Y_1, f_1) and (Y_2, f_2) of X are said to be *equivalent* iff there exists a homeomorphism $h : Y_1 \longrightarrow Y_2$ such that $h \circ f_1 = f_2$). Let $[(Y_i, f_i)] \in$

$\mathcal{L}(X)$, where $i = 1, 2$. We set $[(Y_1, f_1)] \leq [(Y_2, f_2)]$ if there exists a continuous mapping $h : Y_2 \longrightarrow Y_1$ such that $f_1 = h \circ f_2$. Then $(\mathcal{L}(X), \leq)$ is a poset (=partially ordered set).

Let X be a Tychonoff space. We will denote by $\mathcal{K}(X)$ the set of all, up to equivalence, Hausdorff compactifications of X.

1.7 Recall that if X is a set and $P(X)$ is the power set of X ordered by the inclusion, then a triple $(X, \delta, \mathcal{B})$ is called a *local proximity space* (see [10]) if $\mathcal{B}$ is an ideal (possibly non proper) of $P(X)$ and δ is a symmetric binary relation on $P(X)$ satisfying the following conditions:

(P1) $\emptyset(-\delta)A$ for every $A \subseteq X$ ("$-\delta$" means "not δ");

(P2) $A\delta A$ for every $A \neq \emptyset$;

(P3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;

(BC1) If $A \in \mathcal{B}$, $C \subseteq X$ and $A \ll C$ (where, for $D, E \subseteq X$, $D \ll E$ iff $D(-\delta)(X\backslash E)$) then there exists a $B \in \mathcal{B}$ such that $A \ll B \ll C$;

(BC2) If $A\delta C$, then there is a $B \in \mathcal{B}$ such that $B \subseteq C$ and $A\delta B$.

A local proximity space $(X, \delta, \mathcal{B})$ is said to be *separated* if δ is the identity relation on singletons. Recall that every separated local proximity space $(X, \delta, \mathcal{B})$ induces a Tychonoff topology $\tau_{(X,\delta,\mathcal{B})}$ in X by defining $\mathrm{cl}(M) = \{x \in X \mid x\delta M\}$ for every $M \subseteq X$ ([10]). If (X, τ) is a topological space then we say that $(X, \delta, \mathcal{B})$ is a *local proximity space on* (X, τ) if $\tau_{(X,\delta,\mathcal{B})} = \tau$.

The set of all separated local proximity spaces on a Tychonoff space (X, τ) will be denoted by $\mathcal{LP}(X, \tau)$. An order in $\mathcal{LP}(X, \tau)$ is defined by $(X, \beta_1, \mathcal{B}_1) \preceq (X, \beta_2, \mathcal{B}_2)$ if $\beta_2 \subseteq \beta_1$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$ (see [10]).

A function $f : X_1 \longrightarrow X_2$ between two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ is said to be an *equicontinuous mapping* (see [10]) if the following two conditions are fulfilled:

(EQ1) $A\beta_1 B$ implies $f(A)\beta_2 f(B)$, for $A, B \subseteq X$, and

(EQ2) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$.

Theorem 1.8 (S. Leader [10]) *Let (X, τ) be a Tychonoff space. Then there exists an isomorphism Λ_X between the ordered sets $(\mathcal{L}(X, \tau), \leq)$ and $(\mathcal{LP}(X, \tau), \preceq)$. In more details, for every $(X, \rho, \mathcal{B}) \in \mathcal{LP}(X, \tau)$ there exists a locally compact Hausdorff extension (Y, f) of X satisfying the following two conditions:*

(a) $A\rho B$ iff $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(B)) \neq \emptyset$;

(b) $B \in \mathcal{B}$ iff $\mathrm{cl}_Y(f(B))$ is compact.

Such a local compactification is unique up to equivalence; we set $(Y, f) = L(X, \rho, \mathcal{B})$ and $(\Lambda_X)^{-1}(X, \rho, \mathcal{B}) = [(Y, f)]$. The space Y is compact iff $X \in \mathcal{B}$. Conversely, if (Y, f) is a locally compact Hausdorff extension of X and ρ and $\mathcal{B}$ are defined by (a) and (b), then $(X, \rho, \mathcal{B})$ is a separated local proximity space, and we set $\Lambda_X([(Y, f)]) = (X, \rho, \mathcal{B})$.

Let $(X_i, \beta_i, \mathcal{B}_i)$, $i = 1, 2$, be two separated local proximity spaces and $f : X_1 \longrightarrow X_2$ be a function. Let $(Y_i, f_i) = L(X_i, \beta_i, \mathcal{B}_i)$, where $i = 1, 2$. Then there exists a continuous map $L(f) : Y_1 \longrightarrow Y_2$ such that $f_2 \circ f = L(f) \circ f_1$ iff f is an equicontinuous map between $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$.

Recall that a subset F of a topological space (X, τ) is called *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$. Clearly, F is regular closed iff it is the closure of an open set. For any topological space (X, τ), the collection $RC(X, \tau)$ (we will often write simply $RC(X)$) of all regular closed subsets of (X, τ) becomes a complete Boolean algebra $(RC(X, \tau), 0, 1, \wedge, \vee, {}^*)$ under the following operations: $1 = X, 0 = \emptyset, F^* = \mathrm{cl}(X \setminus F), F \vee G = F \cup G, F \wedge G = \mathrm{cl}(\mathrm{int}(F \cap G))$. The infinite operations are given by the following formulas: $\bigvee\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\bigcup\{F_\gamma \mid \gamma \in \Gamma\})$ and $\bigwedge\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\mathrm{int}(\bigcap\{F_\gamma \mid \gamma \in \Gamma\}))$. We denote by $CR(X, \tau)$ the family of all compact regular closed subsets of (X, τ). We will often write $CR(X)$ instead of $CR(X, \tau)$.

We will need a lemma from [3]:

Lemma 1.9 *Let X be a dense subspace of a topological space Y. Then the functions $r : RC(Y) \longrightarrow RC(X)$, $F \mapsto F \cap X$, and $e : RC(X) \longrightarrow RC(Y)$, $G \mapsto \mathrm{cl}_Y(G)$, are Boolean isomorphisms between Boolean algebras $RC(X)$ and $RC(Y)$, and $e \circ r = id_{RC(Y)}$, $r \circ e = id_{RC(X)}$.*

2 A Generalization of Dwinger Theorem

Definition 2.1 Let X be a zero-dimensional Hausdorff space. Then:

(a) A ZLBA (A, I) is called *admissible for* X if A is a Boolean subalgebra of the Boolean algebra $CO(X)$ and I is an open base of X.

(b) The set of all admissible for X ZLBAs is denoted by $\mathcal{ZA}(X)$.

(c) If $(A_1, I_1), (A_2, I_2) \in \mathcal{ZA}(X)$ then we set $(A_1, I_1) \preceq_0 (A_2, I_2)$ if A_1 is a Boolean subalgebra of A_2 and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$.

Notation 2.2 The set of all (up to equivalence) zero-dimensional locally compact Hausdorff extensions of a zero-dimensional Hausdorff space X will be denoted by $\mathcal{L}_0(X)$.

Theorem 2.3 *Let X be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{L}_0(X), \leq)$ and $(\mathcal{ZA}(X), \preceq_0)$ are isomorphic; moreover, the zero-dimensional compact Hausdorff extensions of X correspond to ZLBAs of the form (A, A).*

Proof. Let (Y, f) be a locally compact Hausdorff zero-dimensional extensions of X. Set

(1) $$A_{(Y,f)} = f^{-1}(CO(Y)) \text{ and } I_{(Y,f)} = f^{-1}(CK(Y)).$$

Note that $A_{(Y,f)} = \{F \in CO(X) \mid \mathrm{cl}_Y(f(F)) \text{ is open in } Y\}$ and $I_{(Y,f)} = \{F \in A_{(Y,f)} \mid \mathrm{cl}_Y(f(F)) \text{ is compact}\}$. We will show that $(A_{(Y,f)}, I_{(Y,f)}) \in \mathcal{ZA}(X)$. Obviously, the map $r^0_{(Y,f)} : (CO(Y), CK(Y)) \longrightarrow (A_{(Y,f)}, I_{(Y,f)})$, $G \mapsto f^{-1}(G)$, is a Boolean isomorphism such that $r^0_{(Y,f)}(CK(Y)) = I_{(Y,f)}$. Hence $(A_{(Y,f)}, I_{(Y,f)})$ is a ZLBA and $r^0_{(Y,f)}$ is an LBA-isomorphism. It is easy to see that $I_{(Y,f)}$ is a base of X (because Y is locally compact). Hence $(A_{(Y,f)}, I_{(Y,f)}) \in \mathcal{ZA}(X)$. It is clear that if (Y_1, f_1) is a locally compact Hausdorff zero-dimensional extensions of X equivalent to the extension (Y, f), then $(A_{(Y,f)}, I_{(Y,f)}) = (A_{(Y_1,f_1)}, I_{(Y_1,f_1)})$. Therefore, a map

$$\alpha^0_X : \mathcal{L}_0(X) \longrightarrow \mathcal{ZA}(X),\ [(Y,f)] \mapsto (A_{(Y,f)}, I_{(Y,f)}), \tag{2}$$

is well-defined.

Let $(A, I) \in \mathcal{ZA}(X)$ and $Y = \Theta^a(A, I)$. Then Y is a locally compact Hausdorff zero-dimensional space. For every $x \in X$, set

$$u_{x,A} = \{F \in A \mid x \in F\}. \tag{3}$$

Since I is a base of X, we get that $u_{x,A}$ is an ultrafilter in A and $u_{x,A} \cap I \neq \emptyset$, i.e. $u_{x,A} \in Y$. Define

$$f_{(A,I)} : X \longrightarrow Y,\ x \mapsto u_{x,A}. \tag{4}$$

Set, for short, $f = f_{(A,I)}$. Obviously, $\mathrm{cl}_Y(f(X)) = Y$. It is easy to see that f is a homeomorphic embedding. Hence (Y, f) is a locally compact Hausdorff zero-dimensional extension of X. We now set:

$$\beta^0_X : \mathcal{ZA}(X) \longrightarrow \mathcal{L}_0(X),\ (A, I) \mapsto [(\Theta^a(A, I), f_{(A,I)})]. \tag{5}$$

We will show that $\alpha^0_X \circ \beta^0_X = id_{\mathcal{ZA}(X)}$ and $\beta^0_X \circ \alpha^0_X = id_{\mathcal{L}_0(X)}$.

Let $[(Y, f)] \in \mathcal{L}_0(X)$. Set, for short, $A = A_{(Y,f)}$, $I = I_{(Y,f)}$, $g = f_{(A,I)}$, $Z = \Theta^a(A, I)$ and $\varphi = r^0_{(Y,f)}$. Then $\beta^0_X(\alpha^0_X([(Y, f)])) = \beta_X(A, I) = [(Z, g)]$. We have to show that $[(Y, f)] = [(Z, g)]$. Since φ is an LBA-isomorphism, we get that $h = \Theta^a(\varphi) : Z \longrightarrow \Theta^a(\Theta^t(Y))$ is a homeomorphism. Set $Y' = \Theta^a(\Theta^t(Y))$. By Theorem 1.6, the map $t^C_Y : Y \longrightarrow Y'$, $y \mapsto u^{CO(Y)}_y$ is a homeomorphism. Let $h' = (t^C_Y)^{-1} \circ h$. Then $h' : Z \longrightarrow Y$ is a homeomorphism. We will prove that $h' \circ g = f$ and this will imply that $[(Y, f)] = [(Z, g)]$. Let $x \in X$. Then $h'(g(x)) = h'(u_{x,A}) = (t^C_Y)^{-1}(h(u_{x,A})) = (t_Y)^{-1}(\varphi^{-1}(u_{x,A}))$. We have that $u_{x,A} = \{f^{-1}(F) \mid F \in CO(Y), x \in f^{-1}(F)\} = \{\varphi(F) \mid F \in CO(Y), x \in f^{-1}(F)\}$. Thus $\varphi^{-1}(u_{x,A}) = \{F \in CO(Y) \mid f(x) \in F\} = u^{CO(Y)}_{f(x)}$. Hence $(t_Y)^{-1}(\varphi^{-1}(u_{x,A})) = f(x)$. So, $h' \circ g = f$. Therefore, $\beta^0_X \circ \alpha^0_X = id_{\mathcal{L}_0(X)}$.

Let $(A, I) \in \mathcal{ZA}(X)$ and $Y = \Theta^a(A, I)$. Set $f = f_{(A,I)}$, $B = A_{(Y,f)}$ and $J = I_{(Y,f)}$. Then $\alpha^0_X(\beta^0_X(A, I)) = (B, J)$. By Theorem 1.6, we have that $\lambda^C_A : (A, I) \longrightarrow (CO(Y), CK(Y))$ is an LBA-isomorphism. Hence $\lambda^C_A(A) = CO(Y)$ and $\lambda^C_A(I) = CK(Y)$. We will show that $f^{-1}(\lambda^C_A(F)) = F$, for every $F \in A$. Recall that $\lambda^C_A(F) = \{u \in Y \mid F \in u\}$. Now we have that if $F \in A$ then $f^{-1}(\lambda^C_A(F)) = \{x \in$

$X \mid f(x) \in \lambda_A^C(F)\} = \{x \in X \mid u_{x,A} \in \lambda_A^C(F)\} = \{x \in X \mid F \in u_{x,A}\} = \{x \in X \mid x \in F\} = F$. Thus

$$B = f^{-1}(CO(Y)) = A \text{ and } J = f^{-1}(CK(Y)) = I. \tag{6}$$

Therefore, $\alpha_X^0 \circ \beta_X^0 = id_{\mathcal{ZA}(X)}$.

We will now prove that α_X^0 and β_X^0 are monotone maps.

Let $[(Y_i, f_i)] \in \mathcal{L}_0(X)$, where $i = 1,2$, and $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Then there exists a continuous map $g : Y_2 \longrightarrow Y_1$ such that $g \circ f_2 = f_1$. Set $A_i = A_{(Y_i,f_i)}$ and $I_i = I_{(Y_i,f_i)}$, $i = 1,2$. Then $\alpha_X^0([(Y_i, f_i)]) = (A_i, I_i)$, where $i = 1,2$. We have to show that $A_1 \subseteq A_2$ and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$. Let $F \in A_1$. Then $F' = \mathrm{cl}_{Y_1}(f_1(F)) \in CO(Y_1)$ and, hence, $G' = g^{-1}(F') \in CO(Y_2)$. Thus $(f_2)^{-1}(G') \in A_2$. Since $(f_2)^{-1}(G') = (f_2)^{-1}(g^{-1}(F')) = (f_2)^{-1}(g^{-1}(\mathrm{cl}_{Y_1}(f_1(F)))) = (f_1)^{-1}(\mathrm{cl}_{Y_1}(f_1(F))) = F$, we get that $F \in A_2$. Therefore, $A_1 \subseteq A_2$. Further, let $V \in I_2$. Then $V' = \mathrm{cl}_{Y_2}(f_2(V)) \in CK(Y_2)$. Thus $g(V')$ is a compact subset of Y_1. Hence there exists $U \in I_1$ such that $g(V') \subseteq \mathrm{cl}_{Y_1}(f_1(U))$. Then $V \subseteq (f_2)^{-1}(g^{-1}(g(\mathrm{cl}_{Y_2}(f_2(V))))) = (f_1)^{-1}(g(V')) \subseteq (f_1)^{-1}(\mathrm{cl}_{Y_1}(f_1(U))) = U$. So, $\alpha_X^0([(Y_1, f_1)]) \preceq_0 \alpha_X^0([(Y_2, f_2)])$. Hence, α_X^0 is a monotone function.

Let now $(A_i, I_i) \in \mathcal{ZA}(X)$, where $i = 1,2$, and $(A_1, I_1) \preceq_0 (A_2, I_2)$. Set, for short, $Y_i = \Theta^a(A_i, I_i)$ and $f_i = f_{(A_i,I_i)}$, $i = 1,2$. Then $\beta_X^0(A_i, I_i) = [(Y_i, f_i)]$, $i = 1,2$. We will show that $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. We have that, for $i = 1,2$, $f_i : X \longrightarrow Y_i$ is defined by $f_i(x) = u_{x,A_i}$, for every $x \in X$. We also have that $A_1 \subseteq A_2$ and for every $V \in I_2$ there exists $U \in I_1$ such that $V \subseteq U$. Let us regard the function $\varphi : (A_1, I_1) \longrightarrow (A_2, I_2)$, $F \mapsto F$. Obviously, φ is a **ZLBA**-morphism. Then $g = \Theta^a(\varphi) : Y_2 \longrightarrow Y_1$ is a continuous map. We will prove that $g \circ f_2 = f_1$, i.e. that for every $x \in X$, $g(u_{x,A_2}) = u_{x,A_1}$. So, let $x \in X$. We have that $u_{x,A_2} = \{F \in A_2 \mid x \in F\}$ and $g(u_{x,A_2}) = \varphi^{-1}(u_{x,A_2})$. Clearly, $\varphi^{-1}(u_{x,A_2}) = \{F \in A_1 \cap A_2 \mid x \in F\}$. Since $A_1 \subseteq A_2$, we get that $\varphi^{-1}(u_{x,A_2}) = \{F \in A_1 \mid x \in F\} = u_{x,A_1}$. So, $g \circ f_2 = f_1$. Thus $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Therefore, β_X^0 is also a monotone function. Since $\beta_X^0 = (\alpha_X^0)^{-1}$, we get that α_X^0 (as well as β_X^0) is an isomorphism. □

Definition 2.4 Let X be a zero-dimensional Hausdorff space. A Boolean algebra A is called *admissible for* X (or, a *Boolean base of* X) if A is a Boolean subalgebra of the Boolean algebra $CO(X)$ and A is an open base of X. The set of all admissible Boolean algebras for X is denoted by $\mathcal{BA}(X)$.

Notation 2.5 The set of all (up to equivalence) zero-dimensional compact Hausdorff extensions of a zero-dimensional Hausdorff space X will be denoted by $\mathcal{K}_0(X)$.

Corollary 2.6 (Ph. Dwinger [7]) *Let X be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{K}_0(X), \leq)$ and $(\mathcal{BA}(X), \subseteq)$ are isomorphic.*

Proof. Clearly, a Boolean algebra A is admissible for X iff the ZLBA (A, A) is admissible for X. Also, if A_1, A_2 are two admissible for X Boolean algebras then $A_1 \subseteq A_2$ iff $(A_1, A_1) \preceq_0 (A_2, A_2)$. Since the admissible ZLBAs of the form (A, A) and only they correspond to the zero-dimensional compact Hausdorff extensions of X, it becomes obvious that our assertion follows from Theorem 2.3. □

3 Zero-dimensional Local Proximities

Definition 3.1 A local proximity $(X, \delta, \mathcal{B})$ is called *zero-dimensional* if for every $A, B \in \mathcal{B}$ with $A \ll B$ there exists $C \subseteq X$ such that $A \subseteq C \subseteq B$ and $C \ll C$.

The set of all separated zero-dimensional local proximity spaces on a Tychonoff space (X, τ) will be denoted by $\mathcal{LP}_0(X, \tau)$. The restriction of the order relation $\preceq$ in $\mathcal{LP}(X, \tau)$ (see 1.7) to the set $\mathcal{LP}_0(X, \tau)$ will be denoted again by $\preceq$.

Theorem 3.2 *Let (X, τ) be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{L}_0(X), \leq)$ and $(\mathcal{LP}_0(X, \tau), \preceq)$ are isomorphic (see 3.1 and 2.3 for the notations).*

Proof. Having in mind Leader's Theorem 1.8, we need only to show that if $[(Y, f)] \in \mathcal{L}(X)$ and $\Lambda_X([(Y, f)]) = (X, \delta, \mathcal{B})$ then Y is a zero-dimensional space iff $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$.

So, let Y be a zero-dimensional space. Then, by Theorem 1.8, $\mathcal{B} = \{B \subseteq X \mid \mathrm{cl}_Y(f(B)) \text{ is compact}\}$, and for every $A, B \subseteq X$, $A\delta B$ iff $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(B)) \neq \emptyset$. Let $A, B \in \mathcal{B}$ and $A \ll B$. Then $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(X \setminus B)) = \emptyset$. Since $\mathrm{cl}_Y(f(A))$ is compact and Y is zero-dimensional, there exists $U \in CO(Y)$ such that $\mathrm{cl}_Y(f(A)) \subseteq U \subseteq Y \setminus \mathrm{cl}_Y(f(X \setminus B))$. Set $V = f^{-1}(U)$. Then $A \subseteq V \subseteq \mathrm{int}_X(B)$, $\mathrm{cl}_Y(f(V)) = U$ and $\mathrm{cl}_Y(f(X \setminus V)) = Y \setminus U$. Thus $V \ll V$ and $A \subseteq V \subseteq B$. Therefore, $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$.

Conversely, let $(X, \delta, \mathcal{B}) \in \mathcal{LP}_0(X)$ and $(Y, f) = L(X, \delta, \mathcal{B})$ (see 1.8 for the notations). We will prove that Y is a zero-dimensional space. We have again, by Theorem 1.8, that the formulas written in the preceding paragraph for $\mathcal{B}$ and δ take place. Let $y \in Y$ and U be an open neighborhood of y. Since Y is locally compact and Hausdorff, there exist $F_1, F_2 \in CR(Y)$ such that $y \in F_1 \subseteq \mathrm{int}_Y(F_2) \subseteq F_2 \subseteq U$. Let $A_i = f^{-1}(F_i)$, $i = 1, 2$. Then $\mathrm{cl}_Y(f(A_i)) = F_i$, and hence $A_i \in \mathcal{B}$, for $i = 1, 2$. Also, $A_1 \ll A_2$. Thus there exists $C \in \mathcal{B}$ such that $A_1 \subseteq C \subseteq A_2$ and $C \ll C$. It is easy to see that $F_1 \subseteq \mathrm{cl}_Y(f(C)) \subseteq F_2$ and that $\mathrm{cl}_Y(f(C)) \in CO(Y)$. Therefore, Y is a zero-dimensional space. □

By Theorem 1.8, for every Tychonoff space (X, τ), the local proximities of the form $(X, \delta, P(X))$ on (X, τ) and only they correspond to the Hausdorff compactifications of (X, τ). The pairs (X, δ) for which the triple $(X, \delta, P(X))$ is a local proximity are called *Efremovič proximities*. Hence, Leader's Theorem 1.8 implies the famous Smirnov Compactification Theorem [16]. The notion of a zero-dimensional

proximity was introduced recently by G. Bezhanishvili [2]. Our notion of a zero-dimensional local proximity is a generalization of it. We will denote by $\mathcal{P}_0(X)$ the set of all zero-dimensional proximities on a zero-dimensional Hausdorff space X. Now it becomes clear that our Theorem 3.2 implies immediately the following theorem of G. Bezhanishvili [2]:

Corollary 3.3 (G. Bezhanishvili [2]) *Let (X,τ) be a zero-dimensional Hausdorff space. Then there exists an isomorphism between the ordered sets $(\mathcal{K}_0(X),\leq)$ and $(\mathcal{P}_0(X,\tau),\preceq)$ (see 3.1 and 2.3 for the notations).*

The connection between the zero-dimensional local proximities on a zero-dimensional Hausdorff space X and the admissible for X ZLBAs is clarified in the next result:

Theorem 3.4 *Let (X,τ) be a zero-dimensional Hausdorff space. Then:*

(a) Let $(A,I)\in\mathcal{ZA}(X,\tau)$. Set $\mathcal{B}=\{M\subseteq X \mid \exists B\in I$ such that $M\subseteq B\}$, and for every $M,N\in\mathcal{B}$, let $M\delta N \iff (\forall F\in I)[(M\subseteq F)\rightarrow(F\cap N\neq\emptyset)]$; further, for every $K,L\subseteq X$, let $K\delta L \iff [\exists M,N\in\mathcal{B}$ such that $M\subseteq K, N\subseteq L$ and $M\delta N]$. Then $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$. Set $(X,\delta,\mathcal{B})=L_X(A,I)$.

(b) Let $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$. Set $A=\{F\subseteq X \mid F\ll F\}$ and $I=A\cap\mathcal{B}$. Then $(A,I)\in\mathcal{ZA}(X,\tau)$. Set $(A,I)=l_X(X,\delta,\mathcal{B})$.

(c) $\beta_X^0=(\Lambda_X)^{-1}\circ L_X$ and, for every $(X,\delta,\mathcal{B})\in\mathcal{LP}_0(X,\tau)$, $(\beta_X^0\circ l_X)(X,\delta,\mathcal{B})=(\Lambda_X)^{-1}(X,\delta,\mathcal{B})$ (see 1.8, (5), as well as (a) and (b) here for the notations);

(d) The correspondence $L_X:(\mathcal{ZA}(X,\tau),\preceq_0)\longrightarrow(\mathcal{LP}_0(X,\tau),\preceq)$ is an isomorphism (between posets) and $L_X^{-1}=l_X$.

Proof. It follows from Theorems 2.3, 3.2 and 1.8. ☐

The above assertion is a generalization of the analogous result of G. Bezhanishvili [2] concerning the connection between the zero-dimensional proximities on a zero-dimensional Hausdorff space X and the admissible for X Boolean algebras.

4 Extensions over Zero-dimensional Local Compactifications

Theorem 4.1 *Let (X_i,τ_i), where $i=1,2$, be zero-dimensional Hausdorff spaces, (Y_i,f_i) be a zero-dimensional Hausdorff local compactification of (X_i,τ_i), $(A_i,I_i)=\alpha_X^0(Y_i,f_i)$ (see (2) and (1) for $\alpha_{X_i}^0$)), where $i=1,2$, and $f:X_1\longrightarrow X_2$ be a function. Then there exists a continuous function $g=L_0(f):Y_1\longrightarrow Y_2$ such that $g\circ f_1=f_2\circ f$ iff f satisfies the following conditions:*

(ZEQ1) *For every $G\in A_2$, $f^{-1}(G)\in A_1$ holds;*

(ZEQ2) *For every $F\in I_1$ there exists $G\in I_2$ such that $f(F)\subseteq G$.*

Proof. ($\Rightarrow$) Let there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$. By Lemma 1.9 and (6), we have that the maps

$$(7)\quad r_i^c : CO(Y_i) \longrightarrow A_i,\ G \mapsto (f_i)^{-1}(G),\ e_i^c : A_i \longrightarrow CO(Y_i),\ F \mapsto \mathrm{cl}_{Y_i}(f_i(F)),$$

where $i = 1, 2$, are Boolean isomorphisms; moreover, since $r_i^c(CK(Y_i)) = I_i$ and $e_i^c(I_i) = CK(Y_i)$, we get that

$$(8)\quad r_i^c : (CO(Y_i), CK(Y_i)) \longrightarrow (A_i, I_i) \text{ and } e_i^c : (A_i, I_i) \longrightarrow (CO(Y_i), CK(Y_i)),$$

where $i = 1, 2$, are LBA-isomorphisms. Set

$$(9)\qquad \psi_g : CO(Y_2) \longrightarrow CO(Y_1),\ G \mapsto g^{-1}(G),\ \text{and } \psi_f = r_1^c \circ \psi_g \circ e_2^c.$$

Then $\psi_f : A_2 \longrightarrow A_1$. We will prove that

$$(10)\qquad \psi_f(G) = f^{-1}(G),\ \text{for every } G \in A_2.$$

Indeed, let $G \in A_2$. Then $\psi_f(G) = (r_1^c \circ \psi_g \circ e_2^c)(G) = (f_1)^{-1}(g^{-1}(\mathrm{cl}_{Y_2}(f_2(G)))) = \{x \in X_1 \mid (g \circ f_1)(x) \in \mathrm{cl}_{Y_2}(f_2(G))\} = \{x \in X_1 \mid f_2(f(x)) \in \mathrm{cl}_{Y_2}(f_2(G))\} = \{x \in X_1 \mid f(x) \in (f_2)^{-1}(\mathrm{cl}_{Y_2}(f_2(G)))\} = \{x \in X_1 \mid f(x) \in G\} = f^{-1}(G)$. This shows that condition (ZEQ1) is fulfilled. Since, by Theorem 1.6, $\psi_g = \Theta^t(g)$, we get that ψ_g is a **ZLBA**-morphism. Thus ψ_f is a **ZLBA**-morphism. Therefore, for every $F \in I_1$ there exists $G \in I_2$ such that $f^{-1}(G) \supseteq F$. Hence, condition (ZEQ2) is also checked.
($\Leftarrow$) Let f be a function satisfying conditions (ZEQ1) and (ZEQ2). Set $\psi_f : A_2 \longrightarrow A_1,\ G \mapsto f^{-1}(G)$. Then $\psi_f : (A_2, I_2) \longrightarrow (A_1, I_1)$ is a **ZLBA**-morphism. Put $g = \Theta^a(\psi_f)$. Then $g : \Theta^a(A_1, I_1) \longrightarrow \Theta^a(A_2, I_2)$, i.e. $g : Y_1 \longrightarrow Y_2$ and g is a continuous function (see Theorem 1.6 and (5)). We will show that $g \circ f_1 = f_2 \circ f$. Let $x \in X_1$. Then, by (4) and Theorem 1.6, $g(f_1(x)) = g(u_{x,A_1}) = (\psi_f)^{-1}(u_{x,A_1}) = \{G \in A_2 \mid \psi_f(G) \in u_{x,A_1}\} = \{G \in A_2 \mid x \in f^{-1}(G)\} = \{G \in A_2 \mid f(x) \in G\} = u_{f(x),A_2} = f_2(f(x))$. Thus, $g \circ f_1 = f_2 \circ f$. $\square$

It is natural to write $f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)$ when we have a situation like that which is described in Theorem 4.1. Then, in analogy with the Leader's equicontinuous functions (see Leader's Theorem 1.8), the functions $f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)$ which satisfy conditions (ZEQ1) and (ZEQ2) will be called *0-equicontinuous functions.* Since I_2 is a base of X_2, we obtain that every 0-equcontinuous function is a continuous function.

Corollary 4.2 *Let (X_i, τ_i), $i = 1, 2$, be two zero-dimensional Hausdorff spaces, $A_i \in \mathcal{BA}(X_i)$, $(Y_i, f_i) = \beta^0_{X_i}(A_i, A_i)$ (see (5) for $\beta^0_{X_i}$), where $i = 1, 2$, and $f : X_1 \longrightarrow X_2$ be a function. Then there exists a continuous function $g = L_0(f) : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$ iff f satisfies condition* (ZEQ1).

Proof. It follows from Theorem 4.1 because for ZLBAs of the form (A_i, A_i), where $i = 1, 2$, condition (ZEQ2) is always fulfilled. $\square$

Clearly, Theorem 2.6 implies (and this is noted in [7]) that every zero-dimensional Hausdorff space X has a greatest zero-dimensional Hausdorff compactification which corresponds to the admissible for X Boolean algebra $CO(X)$. This compactification was discovered by B. Banaschewski [1]; it is denoted by $(\beta_0 X, \beta_0)$ and it is called *Banaschewski compactification* of X. One obtains immediately its main property using our Corollary 4.2:

Corollary 4.3 (B. Banaschewski [1]) *Let (X_i, τ_i), $i = 1, 2$, be two zero-dimensional Hausdorff spaces and (cX_2, c) be a zero-dimensional Hausdorff compactification of X_2. Then for every continuous function $f : X_1 \longrightarrow X_2$ there exists a continuous function $g : \beta_0 X_1 \longrightarrow cX_2$ such that $g \circ \beta_0 = c \circ f$.*

Proof. Since $\beta_0 X_1$ corresponds to the admissible for X_1 Boolean algebra $CO(X_1)$, condition (ZEQ1) is clearly fulfilled when f is a continuous function. Now apply Corollary 4.2. □

If in the above Corollary 4.3 $cX_2 = \beta_0 X_2$, then the map g will be denoted by $\beta_0 f$.

Recall that a function $f : X \longrightarrow Y$ is called *skeletal* ([13]) if

$$\mathrm{int}(f^{-1}(\mathrm{cl}(V))) \subseteq \mathrm{cl}(f^{-1}(V)) \tag{11}$$

for every open subset V of Y. Recall also the following result:

Lemma 4.4 ([4]) *A function $f : X \longrightarrow Y$ is skeletal iff $\mathrm{int}(\mathrm{cl}(f(U))) \neq \emptyset$, for every non-empty open subset U of X.*

Lemma 4.5 *A continuous map $f : X \longrightarrow Y$, where X and Y are topological spaces, is skeletal iff for every open subset V of Y such that $\mathrm{cl}_Y(V)$ is open, $\mathrm{cl}_X(f^{-1}(V)) = f^{-1}(\mathrm{cl}_Y(V))$ holds.*

Proof. ($\Rightarrow$) Let f be a skeletal continuous map and V be an open subset of Y such that $\mathrm{cl}_Y(V)$ is open. Let $x \in f^{-1}(\mathrm{cl}_Y(V))$. Then $f(x) \in \mathrm{cl}_Y(V)$. Since f is continuous, there exists an open neighborhood U of x in X such that $f(U) \subseteq \mathrm{cl}_Y(V)$. Suppose that $x \notin \mathrm{cl}_X(f^{-1}(V))$. Then there exists an open neighborhood W of x in X such that $W \subseteq U$ and $W \cap f^{-1}(V) = \emptyset$. We obtain that $\mathrm{cl}_Y(f(W)) \cap V = \emptyset$ and $\mathrm{cl}_Y(f(W)) \subseteq \mathrm{cl}_Y(f(U)) \subseteq \mathrm{cl}_Y(V)$. Since, by Lemma 4.4, $\mathrm{int}_Y(\mathrm{cl}_Y(f(W))) \neq \emptyset$, we get a contradiction. Thus $f^{-1}(\mathrm{cl}_Y(V)) \subseteq \mathrm{cl}_X(f^{-1}(V))$. The converse inclusion follows from the continuity of f. Hence $f^{-1}(\mathrm{cl}_Y(V)) = \mathrm{cl}_X(f^{-1}(V))$.

($\Leftarrow$) Suppose that there exists an open subset U of X such that $\mathrm{int}_Y(\mathrm{cl}_Y(f(U))) = \emptyset$ and $U \neq \emptyset$. Then, clearly, $V = Y \setminus \mathrm{cl}_Y(f(U))$ is an open dense subset of Y. Hence $\mathrm{cl}_Y(V)$ is open in Y. Thus $\mathrm{cl}_X(f^{-1}(V)) = f^{-1}(\mathrm{cl}_Y(V)) = f^{-1}(Y) = X$ holds. Therefore $X = \mathrm{cl}_X(f^{-1}(V)) = \mathrm{cl}_X(f^{-1}(Y \setminus \mathrm{cl}_Y(f(U)))) = \mathrm{cl}_X(X \setminus f^{-1}(\mathrm{cl}_Y(f(U))))$. Since $U \subseteq f^{-1}(\mathrm{cl}_Y(f(U)))$, we get that $X \setminus U \supseteq \mathrm{cl}_X(X \setminus f^{-1}(\mathrm{cl}_Y(f(U)))) = X$, a contradiction. Hence, f is a skeletal map. □

Note that the proof of Lemma 4.5 shows that the following assertion is also true:

Lemma 4.6 *A continuous map $f : X \longrightarrow Y$, where X and Y are topological spaces, is skeletal iff for every open dense subset V of Y, $\mathrm{cl}_X(f^{-1}(V)) = X$ holds.*

Lemma 4.7 *Let (X_i, τ_i), $i = 1, 2$, be two topological spaces, (Y_i, f_i) be some extensions of (X_i, τ_i), $i = 1, 2$, $f : X_1 \longrightarrow X_2$ and $g : Y_1 \longrightarrow Y_2$ be two continuous functions such that $g \circ f_1 = f_2 \circ f$. Then g is skeletal iff f is skeletal.*

Proof. ($\Rightarrow$) Let g be skeletal and V be an open dense subset of X_2. Set $U = Ex_{Y_2}(V)$, i.e. $U = Y_2 \setminus \mathrm{cl}_{Y_2}(f_2(X_2 \setminus V))$. Then U is an open dense subset of Y_2 and $f_2^{-1}(U) = V$. Hence, by Lemma 4.6, $g^{-1}(U)$ is a dense open subset of Y_1. We will prove that $f_1^{-1}(g^{-1}(U)) \subseteq f^{-1}(V)$. Indeed, let $x \in f_1^{-1}(g^{-1}(U))$. Then $g(f_1(x)) \in U$, i.e. $f_2(f(x)) \in U$. Thus $f(x) \in f_2^{-1}(U) = V$. So, $f_1^{-1}(g^{-1}(U)) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is dense in X_1. Therefore, by Lemma 4.6, f is a skeletal map.

($\Leftarrow$) Let f be a skeletal map and U be a dense open subset of Y_2. Set $V = f_2^{-1}(U)$. Then V is an open dense subset of X_2. Thus, by Lemma 4.6, $f^{-1}(V)$ is a dense subset of X_1. We will prove that $f^{-1}(V) \subseteq f_1^{-1}(g^{-1}(U))$. Indeed, let $x \in f^{-1}(V)$. Then $f(x) \in V = f_2^{-1}(U)$. Thus $f_2(f(x)) \in U$, i.e. $g(f_1(x)) \in U$. So, $f^{-1}(V) \subseteq f_1^{-1}(g^{-1}(U))$. This implies that $g^{-1}(U)$ is dense in Y_1. Now, Lemma 4.6 shows that g is a skeletal map. $\square$

We are now ready to prove the following result:

Theorem 4.8 *Let (X_i, τ_i), where $i = 1, 2$, be zero-dimensional Hausdorff spaces. Let, for $i = 1, 2$, (Y_i, f_i) be a zero-dimensional Hausdorff local compactification of (X_i, τ_i), $(A_i, I_i) = \alpha^0_X(Y_i, f_i)$ (see (2) and (1) for $\alpha^0_{X_i}$), $f : (X_1, A_1, I_1) \longrightarrow (X_2, A_2, I_2)$ be a 0-equicontinuous function and $g = L_0(f) : Y_1 \longrightarrow Y_2$ be the continuous function such that $g \circ f_1 = f_2 \circ f$ (its existence is guaranteed by Theorem 4.1). Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(ZO) *For every $F \in I_1$, $\mathrm{cl}_{X_2}(f(F)) \in I_2$ holds;*

(c) g is a perfect map iff f satisfies the following condition:

(ZP) *For every $G \in I_2$, $f^{-1}(G) \in I_1$ holds (i.e., briefly, $f^{-1}(I_2) \subseteq I_1$);*

(d) $\mathrm{cl}_{Y_2}(g(Y_1)) = Y_2$ iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(e) g is an injection iff f satisfies the following condition:

(ZI) *For every $F_1, F_2 \in I_1$ such that $F_1 \cap F_2 = \emptyset$ there exist $G_1, G_2 \in I_2$ with $G_1 \cap G_2 = \emptyset$ and $f(F_i) \subseteq G_i$, $i = 1, 2$;*

(f) g is an open injection iff $I_1 \subseteq f^{-1}(I_2)$ and f satisfies condition (ZO)*;*

(g) g is a closed injection iff $f^{-1}(I_2) = I_1$;

(h) g is a perfect surjection iff f satisfies condition (ZP) *and $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;*

(i) g is a dense embedding iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$ and $I_1 \subseteq f^{-1}(I_2)$.

Proof. Set $\psi_g = \Theta^t(g)$ (see Theorem 1.6). Then $\psi_g : CO(Y_2) \longrightarrow CO(Y_1)$, $G \mapsto g^{-1}(G)$. Set also $\psi_f : A_2 \longrightarrow A_1$, $G \mapsto f^{-1}(G)$. Then, (9), (7) and (10) imply that $\psi_f = r_1^c \circ \psi_g \circ e_2^c$.

(a) It follows from Lemma 4.7.

(b) *First Proof.* Using [6, Theorem 2.8(a)] and (8), we get that the map g is open iff there exists a map $\psi^f : I_1 \longrightarrow I_2$ satisfying the following conditions:

(OZL1) For every $F \in I_1$ and every $G \in I_2$, $(F \cap f^{-1}(G) = \emptyset) \rightarrow (\psi^f(F) \cap G = \emptyset)$;

(OZL2) For every $F \in I_1$, $f^{-1}(\psi^f(F)) \supseteq F$.

Obviously, condition (OZL2) is equivalent to the following one: for every $F \in I_1$, $f(F) \subseteq \psi^f(F)$. We will show that for every $F \in I_1$, $\psi^f(F) \subseteq \mathrm{cl}_{X_2}(f(F))$. Indeed, let $y \in \psi^f(F)$ and suppose that $y \notin \mathrm{cl}_{X_2}(f(F))$. Since I_2 is a base of X_2, there exists a $G \in I_2$ such that $y \in G$ and $G \cap f(F) = \emptyset$. Then $F \cap f^{-1}(G) = \emptyset$ and condition (OZL1) implies that $\psi^f(F) \cap G = \emptyset$. We get that $y \notin \psi^f(F)$, a contradiction. Thus $f(F) \subseteq \psi^f(F) \subseteq \mathrm{cl}_{X_2}(f(F))$. Since $\psi^f(F)$ is a closed set, we obtain that $\psi^f(F) = \mathrm{cl}_{X_2}(f(F))$. Obviously, conditions (OZL1) and (OZL2) are satisfied when $\psi^f(F) = \mathrm{cl}_{X_2}(f(F))$. This implies that g is an open map iff for every $F \in I_1$, $\mathrm{cl}_{X_2}(f(F)) \in I_2$.

Second Proof. We have, by (1), that $I_i = (f_i)^{-1}(CK(Y_i))$, for $i = 1, 2$. Thus, for every $F \in I_i$, where $i \in \{1, 2\}$, we have that $\mathrm{cl}_{Y_i}(f_i(F)) \in CK(Y_i)$.

Let g be an open map and $F \in I_1$. Then, $G = \mathrm{cl}_{Y_1}(f_1(F)) \in CK(Y_1)$. Thus $g(G) \in CK(Y_2)$. Since G is compact, we have that $g(G) = \mathrm{cl}_{Y_2}(g(f_1(F))) = \mathrm{cl}_{Y_2}(f_2(f(F))) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$. Therefore, $\mathrm{cl}_{X_2}(f(F)) = (f_2)^{-1}(g(G))$, i.e. $\mathrm{cl}_{X_2}(f(F)) \in I_2$.

Conversely, let f satisfies condition (ZO). Since $CK(Y_1)$ is an open base of Y_1, for showing that g is an open map, it is enough to prove that for every $G \in CK(Y_1)$, $g(G) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$ holds, where $F = (f_1)^{-1}(G)$ and thus $F \in I_1$. Obviously, $G = \mathrm{cl}_{Y_1}(f_1(F))$. Using again the fact that G is compact, we get that $g(G) = g(\mathrm{cl}_{Y_1}(f_1(F))) = \mathrm{cl}_{Y_2}(g(f_1(F))) = \mathrm{cl}_{Y_2}(f_2(f(F))) = \mathrm{cl}_{Y_2}(f_2(\mathrm{cl}_{X_2}(f(F))))$. So, g is an open map.

(c) Since Y_2 is a locally compact Hausdorff space and $CK(Y_2)$ is a base of Y_2, we get, using the well-known [8, Theorem 3.7.18], that g is a perfect map iff $g^{-1}(G) \in CK(Y_1)$ for every $G \in CK(Y_2)$. Thus g is a perfect map iff $\psi_g(G) \in CK(Y_1)$ for every $G \in CK(Y_2)$. Now, (8) and (9) imply that g is a perfect map $\iff$ $\psi_f(G) \in I_1$ for every $G \in I_2$ $\iff$ f satisfies condition (ZP).

(d) This is obvious.

(e) Having in mind (8) and (9), our assertion follows from [6, Theorem 3.5].

(f) It follows from (b), (8), (9), and [6, Theorem 3.12].

(g) It follows from (c), (8), (9), and [6, Theorem 3.14].

(h) It follows from (c) and (d).

(i) It follows from (d) and [6, Theorem 3.28 and Proposition 3.3]. We will also give a *second proof* of this fact. Obviously, if g is a dense embedding then $g(Y_1)$ is an

open subset of Y_2 (because Y_1 is locally compact); thus g is an open mapping and we can apply (f) and (d). Conversely, if $\mathrm{cl}_{X_2}(f(X_1)) = X_2$ and $I_1 \subseteq f^{-1}(I_2)$, then, by (d), $g(Y_1)$ is a dense subset of Y_2. We will show that f satisfies condition (ZO). Let $F_1 \in I_1$. Then there exists $F_2 \in I_2$ such that $F_1 = f^{-1}(F_2)$. Then, obviously, $\mathrm{cl}_{X_2}(f(F_1)) \subseteq F_2$. Suppose that $G_2 = F_2 \setminus \mathrm{cl}_{X_2}(f(F_1)) \neq \emptyset$. Since G_2 is open, there exists $x_2 \in G_2 \cap f(X_1)$. Then there exists $x_1 \in X_1$ such that $f(x_1) = x_2 \in F_2$. Thus $x_1 \in F_1$, a contradiction. Therefore, $\mathrm{cl}_{X_2}(f(F_1)) = F_2$. Thus, $\mathrm{cl}_{X_2}(f(F_1)) \in I_2$. So, condition (ZO) is fulfilled. Hence, by (b), g is an open map. Now, using (f), we get that g is also an injection. All this shows that g is a dense embedding. □

Recall that a continuous map $f : X \longrightarrow Y$ is called *quasi-open* ([12]) if for every non-empty open subset U of X, $\mathrm{int}(f(U)) \neq \emptyset$ holds. As it is shown in [4], if X is regular and Hausdorff, and $f : X \longrightarrow Y$ is a closed map, then f is quasi-open iff f is skeletal. This fact and Theorem 4.8 imply the following two corollaries:

Corollary 4.9 *Let X_1, X_2 be two zero-dimensional Hausdorff spaces and $f : X_1 \longrightarrow X_2$ be a continuous function. Then:*

(a) $\beta_0 f$ is quasi-open iff f is skeletal;

(b) $\beta_0 f$ is an open map iff f satisfies the following condition:

(ZOB) *For every $F \in CO(X_1)$, $\mathrm{cl}_{X_2}(f(F)) \in CO(X_2)$ holds;*

(c) $\beta_0 f$ is a surjection iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(d) $\beta_0 f$ is an injection iff $f^{-1}(CO(X_2)) = CO(X_1)$.

Corollary 4.10 *Let X_1, X_2 be two zero-dimensional Hausdorff spaces, $f : X_1 \longrightarrow X_2$ be a continuous function, $\mathcal{B}$ be a Boolean algebra admissible for X_2, (cX_2, c) be the Hausdorff zero-dimensional compactification of X_2 corresponding to $\mathcal{B}$ (see Theorems 2.3 and 2.6) and $g : \beta_0 X_1 \longrightarrow cX_2$ be the continuous function such that $g \circ \beta_0 = c \circ f$ (its existence is guaranteed by Theorem 4.3). Then:*

(a) g is quasi-open iff f is skeletal;

(b) g is an open map iff f satisfies the following condition:

(ZOC) *For every $F \in CO(X_1)$, $\mathrm{cl}_{X_2}(f(F)) \in \mathcal{B}$ holds;*

(c) g is a surjection iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(d) g is an injection iff $f^{-1}(\mathcal{B}) = CO(X_1)$.

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